AN INTRODUCTION TO INFINITY TOPOI

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Contents

1. The language of quasi-categories

We will use the of quasi-categories developed by Joyal and Lurie. See for instance the book Higher Topos Theory [Lur09].

Definition 1. A quasi-category is a simplicial set C so that any diagram of the form

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & C \\ \downarrow & & \\ \Delta[n] & & \end{array}$$

for $0 < i < n$ has a lift.

One of the advantage of this model of $(\infty,1)$-category is that, for any two quasi-categories C and D, the simplicial internal hom $[C,D]$ is a quasi-category which represents the "infinity-category of functors from C to D".

Definition 2. An adjunction is the data of functors $L : C \to D$ and $R : D \to C$ together with a morphism $\eta : Id_C \to R \circ L$ in $[C,C]$ so that for any object X of C and any object Y of D, the composition

$$\hom_D(L(X),Y) \to \hom_C(RL(X),R(Y)) \to \hom_C(X,R(Y))$$

is an equivalence of ∞-groupoids.

Remark 1. Lurie first defines an adjunction as a functor $A : G \to \Delta[1]$ which is both a cartesian and a cocartesian fibration.

Definition 3. An object X of an quasi-category C is initial if for any object Y, $\hom_C XY$ is contractible. It is final if for any object Y, $\hom_C YX$ is contractible.

Let K be a simplicial set. We denote respectively by $K^{\triangleleft}$ and $K^{\triangleright}$ the cone and the cocone of K.

Definition 4. let $D : K \to C$ be a diagram in a quasi-category C. If it exists, a colimit of D is an initial object in the quasi-category

$$C_{D/} = [K^{\triangleright}, C] \times_{[K,C]} \{D\}.$$

2. Topos theory

The goal of this section is to recall some topos theory.

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2.1. Grothendieck topology.

Definition 5. A sieve on an object X of a small category A is a subobject of X in the category PSh(A) of presheaves on A.

Definition 6. A Grothendieck topology τ on a small category A is the data of, for any object X, a collection Cov(X) of sieves on X called the covering sieves of X, so that

- (Base change) for any morphism $f : X \to Y$ and any $S \in \mathrm{Cov}(X)$, the pullback f^*S of S along f in the category PSh(A) is a covering sieve of Y;
- (Local character) let S be a covering sieve of X and let T be any sieve of X ; if for any $Y \in A$ and any morphism $f \in S(Y) \subset \hom_A(Y, X)$ f^*T is a covering sieve of Y, then T is a covering sieve of X;
- (Identity) The sieve X is a covering sieve of X.

A site (A, τ) is the data of a small category A and of a Grothendieck topology τ on A.

Definition 7. Let (A, τ) be a site. A sheaf on A is a presheaf $F \in \mathrm{PSh}(A)$ so that for any covering sieve U of $X \in A$ the map

$$F(X) = \hom_{\mathrm{PSh}(A)}(X, F) \to \hom_{\mathrm{PSh}(A)}(U, F)$$

is an isomorphism.

Definition 8. Let F be a presheaf on a site (A, τ). Then, the plus construction $F^\dagger$ of F is the presheaf defined by

$$F^\dagger(X) = \mathrm{colim}_{S \to X} F(S)$$

where the colimit is taken over the poset of covering sieves of X. This defines the endofunctor $\dagger$ of PSh(A).

For a presheaf, $F^\dagger$ is not in general a sheaf but only a separated presheaf, meaning that the map $F^\dagger(X) \to \hom_{\mathrm{PSh}(A)}\left(U, F^\dagger\right)$ is a monomorphism for any covering sieve U of X. However $F^{\dagger\dagger}$ is a sheaf. This is actually the closest sheaf to F.

Proposition 1. *The functor $\dagger \circ \dagger$ with values in sheaves is left adjoint to the inclusion functor form sheaves to presheaves.*

2.2. Left exact localisations of a presheaves category.

Definition 9. A left exact localisation of a presheaf category is the data of a category C together with a small category A and an adjunction

$$\mathrm{PSh}(A) \underset{R}{\overset{L}{\rightleftarrows}} C$$

so that R is fully faithful and L is left exact, that is commutes with finite limits.

Proposition 2. *In the adjunction above, the functor i is accessble. Hence,* C *is in particular an accessible localisation of a presheaves category; that is a presentable category.*

2.3. Giraud axioms.

Definition 10. A category C satisfies the Giraud's axioms if

(1) C is presentable;
(2) colimits in C are universal;
(3) unions are disjoint;
(4) equivalence relations are effective.

Some of the points above need some explanation.

- The point (2) means that for any morphism $f : S \to T$, the functor $- \times_T S$ from $C_{/T}$ to $C_{/S}$ preserves colimits. Heuristically, one may think of the bifunctor $- \times_T -$ as a product and of colimits as sums. Hence, this conditions corresponds to the bilinearity of the product.

- The point (3) means that for any two objects, the square

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \sqcup Y. \end{array}$$

- The point (4) needs more explanation.

Definition 11. An equivalence relation in a category C is the data of two objects X and R together with a morphims $R \to X \times X$ so that, for any object Y, the function

$$\hom_C(Y,R) \to \hom_C(Y, X \times X) = \hom_C(Y,X) \times \hom_C(Y,X)$$

defines functorially an equivalence relation on the set $\hom_C(Y,X)$.

Definition 12. An equivalence relation is said to be effective of the morphism $R \to X \times_{X/R} X$ is an isomorphism.

2.4. Definition of a topos.

Definition 13. A topos is a category equivalent to the category of sheaves on a site (A, τ).

Theorem 1. *A category is a topos if and only if it is a left exact localisation of a presheaves category.*

Remark 2. This means in particular that the functor $\dagger \circ \dagger$ is left exact. One may think that $\dagger$ is left exact. But that would implies that any category of separated presheaves is a topos.

Theorem 2. *A category is a topos if and only if it satisfies Giraud's axioms.*

Definition 14. Let T and T′ be two topoi. A geometric morphism f from T to T′ is a functor

$$f^* T' \to T$$

which preserves colimits and is left exact. Hence, it has a right adjoint usually denoted f_*.

Definition 15. A point of a topos T is a geometric morphism

$$x : \mathsf{Set} \to T.$$

One says that T has enough points if for any morphism $f : X \to Y$ in T, the two following conditions are equivalent

- f is an isomorphism;
- for any point x of T, the function $x^*(f)$ is bijective.

3. From topos theory to infinity-topos theory

In this section, we generalise the definition given above the context of infinity categories.

Notation. From now on, for any small ∞-category A, PSh(A) will denote the ∞-category

$$\mathrm{PSh}(A) = [A^{op}, S],$$

where S is the ∞-category of ∞-groupoids.

3.1. Grothendieck topology.

Definition 16. A monomorphism in an ∞-category C is a morphism $f : X \to Y$ so that for any object $Z \in C$ the morphism of ∞-groupoids

$$\hom_C(Z, f) : \hom_C(Z,X) \to \hom_C(Z,Y)$$

is an equivalence on the connected components and induces an injection between the connected components.

Given that definition of a monomorphism, the definition of a Grothendieck topology extends easily to the "Higher context":

- A sieve of an object X of a small ∞-category A is a presheaf $U \in \mathrm{PSh}(A)$ together with a monomorphism $U \to X$.
- A Grothendieck topology is the data of collections of covering sieves on any objects $X \in A$ that satisfies the base change axiom, the local character axiom and the identity axiom.

- A ∞-site is a small ∞-category equipped with a Grothendieck topology.
- A sheaf on an ∞-site (A,τ) is a presheaf F so that for any object X and any covering sieve U of X, the morphism

$$F(X) = \hom_{\mathsf{PSh(A)}}(X,F) \to \hom_{\mathsf{PSh(A)}}(U,F)$$

is an equivalence.

Proposition 3. *The set of Grothendieck topologies on* A *is in bijection with the set of Grothendieck topologies on* $\mathrm{Ho}(\mathsf{A})$.

Idea of the proof. A sieve of an object X may also be described a a full subcategory U of $\mathsf{A}_{/X}$ so that, for any morphism $B \to B'$ in $\mathsf{A}_{/X}$, if B' is in U, then B is U. ☐

3.2. **Higher Giraud's axiom.**

Definition 17. An ∞-category C satisfies the higher Giraud's axioms if

(1) C is presentable;
(2) colimits in C are universal;
(3) unions are disjoint;
(4) groupoids are effective.

The points (1), (2) and (3) are straightforward generalisations of the 1-categorical setting. Let us explain the meaning of the point (4).

Definition 18. We will say that a simplicial object $U : \Delta^{op} \to \mathsf{C}$ in an ∞-category C is a groupoid object if, for any decomposition $[n] = S \cup S'$ with $S \cap S' = \{s\}$, the following diagram is a pullback

$$\begin{array}{ccc} U([n]) & \longrightarrow & U(S) \\ \downarrow & & \downarrow \\ U(S') & \longrightarrow & U(s). \end{array}$$

In particular we get a composition

$$U([1]) \times_{U([0])} U([1]) \simeq U([2]) \xrightarrow{d_1} U([1]).$$

Moreover, we get left inverses

$$U([1]) \simeq U(1<2) \times_{U(2)} U(2) \to U(1<2) \times_{U(2)} U(0<2) \simeq U(0<1<2) \to U(0<1).$$

Definition 19. We will say that an augmented simplicial object $U^+ : \Delta_+^{op} \to \mathsf{C}$ in an ∞-category C is a Cech nerve if, its restriction to Δ^{op} is a groupoid object and if the following diagram is a pullback

$$\begin{array}{ccc} U^+([1]) & \longrightarrow & U^+([0]) \\ \downarrow & & \downarrow \\ U^+([0]) & \longrightarrow & U^+([-1]). \end{array}$$

Lemma 1. *The functor* U^+ *is a Ceck nerve if and only if it is a right Kan extension of its restriction to the full subcategory spanned by* $[-1]$ *and* $[0]$.

Definition 20. A groupoid object in C is effective if it extends Δ_+^{op} by colimit to a Cech nerve .

3.3. **Higher topoi.** The easiest way to extend to the infinity-categorical context the definition of a topos is to extend the "left exact localisation" definition.

Definition 21. An ∞-topos C is an accessible left exact localization of a presheaves category. In other there exists an small category A and an adjunction

$$\mathsf{PSh(A)} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{C}$$

so that R is fully faithful and L is left exact and accessible.

Remark 3. Notice that there is a an additional condition compared to the 1-categorical world: R needs to be accessible. This ensures a topos to be presentable and was a proposition in this world.

Definition 22. Let T and T′ be two ∞-topoi. A geometric morphism f from T to T′ is a functor

$$f^* : \mathsf{T}' \to \mathsf{T}$$

which preserves colimits and is left exact. Hence, it has a right adjoint usually denoted f_*.

Theorem 3. *A category* C *is a topos if and only if it satisfies higher Giraud's axioms.*

To compare ∞-topoi to categories of sheaves on a ∞-site, we will need to study with more details accessible left exact localisations. We will see that such categories of sheaves corresponds to a sub class of such localisations called topological localisations.

4. Reflective subcategories

4.1. Local objects and morphisms.

Definition 23. A reflective subcategory of an ∞-category C is an ∞-adjunction

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}$$

so that R is a full faithful embedding.

The main idea underlying the treatment of reflective subcategories is that it is determined by the set of morphisms $f \in \mathsf{D}_1$ so that $L(f)$ is an equivalence.

Definition 24. Let $S \subset \mathsf{C}_1$ be a set of morphisms of an ∞-category C. Then we say that an object X of C is a S-local if for any morphism $f : U \to V$ in S the map

$$\hom_{\mathsf{C}}(f, X) : \hom_{\mathsf{C}}(V, X) \to \hom_{\mathsf{C}}(U, X)$$

is an equivalence of ∞-groupoids. We denote by $S-\mathrm{loc}$ the set of S-local objects.

Definition 25. A morphism $f : U \to V$ if C is a S-equivalence if for any S-local object X, the map

$$\hom_{\mathsf{C}}(f, X) : \hom_{\mathsf{C}}(V, X) \to \hom_{\mathsf{C}}(U, X)$$

is an equivalence of ∞-groupoids. We denote by $S-\mathrm{eq}$ the set of S-equivalences.

In particular, $S \subset S-\mathrm{eq}$.

Proposition 4. *Consider a reflective subcategory* C *of an ∞-category* D*. Let* $S = L^{-1}(\mathrm{eq})$*. Then the functor* $R : \mathsf{C} \to \mathsf{D}$ *induces an equivalence between* C *and the full subcategory of* D *on* S*-local objects. Moreover, any* S*-equivalence is in* S.

Proof. It is clear that an object in the image of R is S-local. Conversely, for any object X of D, the counit map $\eta(X) : X \to i \circ a(X)$ is in S. If X is S-local, then the map

$$\hom_{\mathsf{D}}(\eta(X), X) : \hom_{\mathsf{D}}(R \circ L(X), X) \to \hom_{\mathsf{D}}(X, X)$$

is an equivalence. Let us choose $f : R \circ L(X) \to X$ so that $f \circ \eta(X) \sim \mathrm{Id}_X$. Since f is also in S, and since $R \circ L(X)$ is S-local then the map, applying

$$\hom_{\mathsf{D}}(f, R \circ L(X)) : \hom_{\mathsf{D}}(X, R \circ L(X)) \to \hom_{\mathsf{D}}(R \circ L(X), R \circ L(X))$$

is an equivalence. Let us choose $g : X \to R \circ L(X)$ so that $g \circ f \sim \mathrm{Id}_{R \circ L(X)}$. Then

$$g \sim g \circ f \circ \eta(X) \sim \eta(X).$$

So f is inverse to $\eta(X)$. So X is equivalent to $R \circ L(X)$ and so is in the essential image of R. □

Corollary 1. *If* D *is continuous, then so is* C.

Proof. The limit of a diagram of S-local objects is S-local. □

Proposition 5. *If* D *is cocontinuous, then so is* C.

Proof. Consider a diagram $D : I \to \mathsf{C}$. Then, we have a sequence of equivalences

$$\begin{aligned}\hom_{\mathrm{Fun}(I,\mathsf{C})}(D, X) \simeq \hom_{\mathrm{Fun}(I,\mathsf{D})}(R \circ D, R(X)) &\simeq \hom_{\mathsf{D}}(\mathrm{colim}(R \circ D), R(X)) \\ &\simeq \hom_{\mathsf{C}}(L(\mathrm{colim}(R \circ D)), X).\end{aligned}$$

This shows that $L(\mathrm{colim}(R \circ D)) = \mathrm{colim} D$. □

4.2. **Strongly saturated set of morphisms.**

Definition 26. A set S of morphisms in an ∞-category D is strongly saturated if

- it is stable under colimits in $[\Delta[1], \mathsf{D}]$;
- it is stable under pushout in D;
- it satisfies the 2-out-of-3 rule.

Definition 27. Let $S \subset D_1$ be a set of morphisms. The strongly saturated set of morphisms $\overline{S}$ generated by S is the smallest strongly saturated set of morphisms that contains S.

Proposition 6. [Lur09, 5.5.4.15] *Let* S *be a small set of morphisms of a presheaves category* $\mathsf{D} = \mathsf{PSh}(\mathsf{A})$, *and let* $R : \mathsf{C} \to \mathsf{D}$ *the full subcategory of* S*-local objects. Then,* R *has a left adjoint* L*. Moreover,*

$$\overline{S} = S - \mathrm{eq} = L^{-1}(\mathrm{eq}).$$

Idea of the proof. The main ingredient of the proof is to build, for any object X of D, a morphism $f : X \to Y$ in $\overline{S}$ so that Y is S-local.

Finally, let us show that a morphism $f : X \to Y$ in $L^{-1}(\mathrm{eq})$ is also in $\overline{S}$. Consider the following square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ RL(X) & \xrightarrow[RL(f)]{} & RL(Y) \end{array}$$

Since the vertical arrows belong to $\overline{S}$ and since the bottom arrow is an equivalence, hence belongs to $\overline{S}$, then f belongs to $\overline{S}$ by the 2-out-of-3 rule. □

4.3. **Additional conditions.**

In this section, we consider a category of presheaves $\mathsf{C} = \mathsf{PSh}(\mathsf{A})$ and a reflective subcategory

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}.$$

Moreover, we denote $S = L^{-1}(\mathrm{eq})$.

Proposition 7. *The functor* L *is preserves finite limits if and only if* S *is stable under pullback.*

Sketch of the proof. If L preserves limits, then it is clear that S is stable under pullback. Conversely, suppose that S is stable under pullbacks. Since the final object of D is S-local, then L preserves the final object. Moreover, let us consider a span $X \to Y \leftarrow Z$. We can write the morphism $X \times_Y Z \to RL(X) \times_{RL(Y)} RL(Z)$ as the composition

$$X \times_Y Z \to X \times_{RL(Y)} Z \to X \times_{RL(Y)} RL(Z) \to RL(X) \times_{RL(Y)} RL(Z).$$

The two last maps are pullbacks of elements of S so are in S. The first map is a pullback of the diagonal map $Y \to Y \times_{RL(Y)} Y$ which has a left inverse given by the projection on the first factor. This projection is in S as a pullback of $Y \to RL(Y)$. So the morphism $X \times_Y Z \to RL(X) \times_{RL(Y)} RL(Z)$ is in S which shows that

$$RL(X \times_Y Z) \simeq RL(X) \times_{RL(Y)} RL(Z).$$

□

Proposition 8 (5.5.1.2 and 5.5.4.2)**.** *The following conditions are equivalent*

- *the functor* R *is accessible (hence* D *is presentable);*
- $S = S_0 - \mathrm{eq}$ *for a small set* S_0.

In this context, we have moreover, $S = \overline{S}_0$.

4.4. **Topological localisation and sheaves topoi.**

Definition 28. A reflective subcategory $L \dashv R$ is called topological if it S is stable under pullbacks and $S = \overline{S_0}$ for a small set S_0 of monomorphisms.

Corollary 2. *A topological localisation is accessible.*

Proposition 9. *Let* (A, τ) *be an* ∞*-site. Then, the inclusion of sheaves into presheaves is a topologocal reflective subcategory. Moreover, this induces a bijection between Grothendieck topologies on* A *and topologocal reflective subcategories of* $\mathrm{PSh}(A)$*.*

Proof. Consider a Grothendieck topology on A. Then, the sheaves are just the S_0-local objects where S_0 is the set of monomorphisms $U \to X$ for any object X of A and any covering sieve U of X. Then, by Proposition 6, the inclusion of sheaves into presheaves is a reflective subcategory. It is clear that it is topological. This gives us a function from Grothendieck topologies to topological localisations. It is injective (any Grothendieck topology is determined by its set of sheaves). Let us show that it is surjective. Consider a topologocal localisation S of $\mathrm{PSh}(A)$ and a set of monomorphisms S_0 so that $S = \overline{S_0}$. Let $f : F \to G$ be a morphism in S_0. For any object X in A and any morphism $X \to G$, the morphism $X \times_G F \to X$ is a monomorphism. As G is a colimit of $A_{/G}$, one can show that f is the colimit of the diagram

$$\begin{aligned} A_{/G} &\to [\Delta[1], \mathrm{PSh}(A)] \\ X &\mapsto (X \times_G F \to X). \end{aligned}$$

Such morphism $X \times_G F \to X$ for any $f \in S_0$ and any $X \in A_{/G}$ gives us the basis of a Grothendieck topology whose sheaves will be exactly the S-local presheaves. □

5. Hypercomplete topoi

5.1. **Effective epimorphisms.** In this section, we are working inside a topos T.

Definition 29. The Cech nerve of a morphism $f : X \to Y$ is the Cech nerve $C(f) : \Delta_+^{op} \to T$ given by

$$\begin{cases} C(f)([n]) = X \times_Y X \times_Y \cdots \times_Y X, \\ C(f)([n-1]) = Y. \end{cases}$$

Definition 30. An effective epimorphism is a morphism f so that $C(f)$ is the extension by colimit of its restriction to Δ_+^{op}.

Notice that there is a one to one correspondance between groupoids (that are effective) and effective epimorphisms.

5.2. **Homotopy groups.** Let A be a small ∞-category and let F be a presheaf on A.

Definition 31. The n^{th}-homotopy group of F is the presheaf on A given by

$$A^{op} \xrightarrow{F} S \qquad\qquad \to \mathrm{Set}$$

Definition 32. $\pi_n(X) = \tau_0(X^{S_n} \to X)$. $\pi_n(f)$ is the π_n of the object $f : X \to Y$ in the ∞-category $T_{/Y}$.

Definition 33. An morphism is ∞-connective if it is an effective epimorphism and if $\pi_n(f) \simeq *$ for any n.

5.3. **Hypercomplete topos.**

Definition 34. An object of a topos T is called hypercomplete if it is local with respect to ∞-connected morphisms. We denote by $T^\wedge$ the full subcategory of hypercomplete objects.

Lemma 2. *The set of* ∞*-connected morphisms is strongly saturated, of small generation and stable under pullbacks.*

Idea of the proof. The full subcategory of $[\Delta[1], T]$ spanned by ∞-connected morphisms is presentable. □

Corollary 3. *The inclusion* $T^\wedge \to T$ *is part of a left exact accessible localisation. Hence,* $T^\wedge$ *is an* ∞*-topos.*

The construction $T \to T^\wedge$ is also functorial.

5.4. **Hypercoverings.**

Definition 35. A simplicial object U on a topos T is an hypercovering if for any natural integer n, the map

$$U_n \to \mathrm{cosk}_{\leq n-1} U$$

is an effective epimoprhism.

References

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

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