Reminders on model categories

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1 Introduction

The theory of model category was introduced by Quillen. It allows to build derived functors in the context of non-abelian categories and to give an effective construction of the localization of a category.

This talk relies mostly on the papers of Dwyer-Spalinski [DS] "Homotopy theories and model categories", on the book "Model categories and their localization" by Philip S. Hirschhorn [Hi] and on Mauro Porta's and my work in the seminar "Autour de la géométrie algébrique dérivée".

2 Model category

In this first section I introduce the first notions on model categories. The proofs can be found in [DS, 3]

Definition 2.1. A model is a usual category M where are distinguished three classes of morphisms: the weak equivalences $\mathfrak{W}$, the cofibrations $\mathfrak{Cof}$ and the fibrations $\mathfrak{Fib}$ which are stable through composition, contains the identities and are such that:

M1 the category M has finite limits and finite colimits.

M2 if f and g are composable morphisms of M, then, if two of the three morphisms f, g an fg are weak equivalences, then also is the third.

M3 a retract of a weak equivalence (resp. cofibration, fibration) is a weak equivalence (resp. cofibration, fibration

M4 Given a commutative square:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[i]{} & D \end{array}$$

there exists a lift

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ C & \xrightarrow[i]{} & D \end{array}$$

whenever h is a cofibration and g is a fibration and one of the two is also a weak equivalence.

M5 A morphism f can be factored in two ways $f = pi$:

(a) i is a cofibration and p an acyclic fibration (i.e an element of $\mathfrak{W} \cap \mathfrak{Fib}$)

(b) i is an acyclic cofibration (i.e an element of $\mathfrak{W} \cap \mathfrak{Cof}$) ans p a fibration

Remark 2.1. An object A is said fibrant if $A \to *$ is a fibration ($*$ is the final object). An object is said cofibrant if $\emptyset \to A$ is cofibrant.

Definition 2.2. A cofibrant replacement of an object A is the data of a factorization $\emptyset \to QA \to A$ of $\emptyset \to A$ where the left map is a cofibration and the right map is a weak equivalence.
A fibrant replacement of A is a factorization $A \to RA \to *$ where the left map is a weak equivalence and the right map is a fibration.

Proposition 2.1 (DS,3)**.** *Let* M *a model category; then*

1. $\mathfrak{Cofib}$ *is exactly the arrows which have the left lifting property with acyclic fibrations*

2. $\mathfrak{Cofib} \cap \mathfrak{W}$ *is exactly the arrows which have the left lifting property with fibrations*

3. $\mathfrak{Fib}$ *is exactly the arrows which have the right lifting property with acyclic cofibrations*

4. $\mathfrak{Fib} \cap \mathfrak{W}$ *is exactly the arrows which have the right lifting property with cofibrations*

Proposition 2.2 (DS,3)**.** *In a model category, fibrations are stable through pullback.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[i]{} & D \end{array}$$

In other words, if i is a fibration then also is f. Acyclic fibrations have the same property. Furthermore, cofibrations an acyclic cofibrations are stable through pushout.

We introduce now one of the most known model categories.

Example 2.1.

1. If A is an abelian category, let $\mathrm{Ch}_{A,\geq 0}$ the category of nonnegatively graded cochain complexes of A. It has a model structure defined by:
   (a) the weak equivalences are the quasi-isomorphisms
   (b) the cofibrations are the maps which are monomorphisms in each degre
   (c) the fibrations are the maps which are epimorphisms in each degree and with injective kernel

2. The category of nonnegatively graded A-chain complexes. It has a model structure defined by:
   (a) the weak equivalences are the quasi-isomorphisms
   (b) the cofibrations are the maps which are monomorphisms in each degree and with projective cokernel
   (c) the fibrations are the maps which are epimorphisms in each degree

3. The category of topological spaces and continous maps Top can be given a model structure:
   (a) the weak equivalences are the weak homotopy equivalences
   (b) the cofibrations are the retract of maps $X \to Y$ where Y is a CW-complex.
   (c) the fibrations are the Serre fibrations

4. The category of simplicial sets can be given a model structure such that:
   (a) the weak equivalences are the weak homotopy equivalences
   (b) the fibrations are the Kan fibrations

3 Homotopy

In this section, M is a model category.

3.1 Left homotopy

Definition 3.1. Let A be an object of M. One calls cylinder object of A the data of a composition of morphisms:

$$A\bigsqcup A \to C_A \to A \tag{1}$$

which factors $id_A + id_A$ and such that the right map is a weak equivalence. It is said :

1. good if $A\bigsqcup A \to C_A$ is a cofibration.
2. very good if it is good and $C_A \to A$ is an (acyclic) fibration.

Definition 3.2. Let $f : A \to B$ be an arrow of a model category M. A left homotopy from f to g is the data of a cylinder object of A and a map $H : C_A \to B$ which factors $f + g : A\bigsqcup A \to B$. The homotopy is said good (resp. very good) if the corresponding cylinder object is.

Proposition 3.1 (DS,4)**.** *If A (with the previous notations) is cofibrant, then the left homotopy is an equivalence relation on $hom_{\mathrm{M}}(A, B)$. We note $\sim^l$ this relation and $\pi^l(hom(A, B))$ the set of equivalence classes.*

Proposition 3.2 (DS,4)**.** *If X is fibrant, then the composition in* M *induces a map:*

$$\pi^l(hom(A, B)) \times \pi^l(hom(B, X)) \to \pi^l(hom(A, X)) \tag{2}$$

3.2 Right homotopy

Definition 3.3. Let A be an object of M. One calls path object of A the data of a composition of morphisms:

$$A \to P_A \to A \times A \tag{3}$$

which factors $id_A \times id_A$ and such that the left map is a weak equivalence. It is said :

1. good if $P_A \to A\times$ is a fibration.
2. very good if it is good and $A \to P_A$ is an (acyclic) cofibration.

Definition 3.4. Let $f : A \to B$ be an arrow of a model category M. A right homotopy from f to g is the data of a path object of B and a map $H : A \to P_B$ which factors $f \times g : A \to B \times B$. The homotopy is said good (resp. very good) if the corresponding path object is.

One has the same kind of the results as in the case of left homotopy.

3.3 Relationship between left and right homotopies

Proposition 3.3. *Let $f, g : X \to Y$ morphisms of* M.

1. *If X is cofibrant an $f \sim^l g$, then $f \sim^r g$.*
2. *If Y is fibrant and $f \sim^r g$, then $f \sim^l g$.*

4 Homotopy category

Definition 4.1. (proposition)[DS,5] From a model category M, one can construct a category $Ho(\mathfrak{M})$ such that.

1. The objects of $Ho(\mathrm{M})$ are those of M
2. For two objects X, Y:

$$hom_{Ho(\mathrm{M})}(X, Y) = \pi hom_{\mathrm{M}_{cf}}(RQX, RQY) \tag{4}$$

   where RQX is a fibrant-cofibrant replacement of X, M_{cf} is the full category of M spanned by the fibrant-cofibrant objects and π is the projection on homotopy class.

Furthermore there is a functor $\gamma : \mathrm{M} \to Ho(\mathrm{M})$ which is the identity on object and sends a morphism to "its homotopy class".

Proposition 4.1 (DS,6)**.** *A morphism f of* M *is a weak equivalence iff $\gamma(f)$ is an isomorphism and the functor $\gamma : \mathrm{M} \to Ho(\mathrm{M})$ is a localization of* M *with respect to $\mathfrak{W}$.*

5 Derived functors

Definition 5.1. Let M be a model category and $F : \mathrm{M} \to \mathrm{C}$ a functor.

1. A left derived functor of F (if it exists) is a pair (LF, t) where LF is a functor $Ho(\mathrm{M}) \to \mathrm{D}$ and t a natural transformation $LF\gamma \to F$ which is universal, ie, for every pair (G, s) of this type, there exists a unique natural transformation $s' : G\gamma \to LF\gamma$ such that $s = t \circ s'$.

2. A right derived functor is an analogous object.

Proposition 5.1 (DS,9)**.** *If one uses the notations of the above definition; if $F(f)$ is an isomorphism whenever f is a weak equivalence between cofibrant objects, then a left derived functor of F exists and for every cofibrant objects X in* M*, the map $t(X) : LF(X) \to F(X)$ is an isomorphism. Similarly, if $F(f)$ is an isomorphism whenever f is a weak equivalence between fibrant objects, then a right derived functor of F exists and for every fibrant objects X in* M*, the map $s(X) : F(X) \to RF(X)$ is an isomorphism.*

Definition 5.2. A total left (right) derived functor of $F : \mathrm{M} \to \mathrm{C}$, where C is a model category, is a left (right) derived functor of $\gamma_C \circ F$.

Example 5.1. Cohomology of sheaves: consider the $\mathrm{C} - \mathrm{Sh}$, the category of non-negatively graded cochains of abelian sheaves on a space X, $C - A$ the category non-negatively graded cochains of abelian groups (both category with the model structure defined above). Consider also the functor $\Gamma : \mathrm{C} - \mathrm{Sh} \to C - A$ which corresponds in each degree to the evaluation of the sheaf on X. The total right derived functor $R_t\Gamma$ exists. Furthermore, if $\mathcal{F}$ is a sheaf consider as a cochain of sheaves concentrated in degree 0 and if I is a fibrant replacement of $\mathcal{F}$ (in $\mathrm{C} - \mathrm{Sh}$), then $R_t\Gamma(\mathcal{F})R_t\Gamma(I) \simeq \mathcal{I}$ in $Ho(C - A)$. This is the same notion as the right derived functor in the context of sheaves cohomology.

Proposition 5.2 (DS,9)**.** *Let* M *and* D *be model categories and $F : \mathrm{M} \rightleftharpoons \mathrm{D} : G$ a pair of djoint functors.*

1. *If F preserves cofibrations and G preserves fibrations, then the total derived functors LF and RG exist and are adjoint.*

2. *If in addition, for every cofibrant object A of* M *and fibrant object of* D *B, a map $A \to G(B)$ is a weak equivalence iff its adjoint $F(A) \to X$ is aweak equivalence, then LF and RG are equivalences of categories.*

Example 5.2. The adjunction $\mathrm{sSet} \rightleftharpoons \mathrm{Top}$ satisfy the conditions and leads to the equivalence $Ho(\mathrm{Top}) \simeq Ho(\mathrm{sSet})$

6 Homotopy limits and colimits

[TV, 2.4]

Definition 6.1. Let C be a (usual) category, and W a sub-category. If I is a (small) category, let C^I be the category of functors $I \to \mathsf{C}$. Then let W_I be the sub-category of C^I spanned by the natural transformations whom morphisms on objects are in W. Then, the constant functor $\mathsf{C} \to \mathsf{C}^I$ induces a functor $\mathsf{C} \to \mathsf{C}^I[(\mathsf{W}_I)^{-1}]$ which sends the morphisms of W to isomorphisms. So, it can be factorized by the projection $\mathsf{C} \to \mathsf{C}[\mathsf{W}^{-1}]$ and the functor:

$$c : \mathsf{C}[\mathsf{W}^{-1}] \to \mathsf{C}^I[(\mathsf{W}_I)^{-1}] \tag{5}$$

1. The homotopy limit of an object A de $\mathsf{C}^I[(\mathsf{W}_I)^{-1}]$ is (if it exists) the data of an object $holim_I(A)$ of $\mathsf{C}[\mathsf{W}^{-1}]$ and isomorphisms of functors:

$$hom_{\mathsf{C}[\mathsf{W}^{-1}]}(., holim_I(A)) \simeq hom_{\mathsf{C}^I[(\mathsf{W}_I)^{-1}]}(c(.), A). \tag{6}$$

   The naturality in A of this isomorphism of functors makes us define the homotopy limit (if it exists) as the right adjoint of c.

2. One defines *hocolim* in an analogous way. The functor *hocolim* is left adjoint to c (if it exists).

7 Mapping spaces

The localization of a model category glues together homotopical maps: therefore one looses the higher homotopical data. The mapping spaces are an attempt to built it.

This section relies on the article "Function complexes in Homotopical Algebra" by W.G. Dwyer and D. Kan.

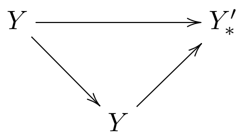
Let M be a model category, and W a the subcategory of weak equivalence.

Definition 7.1. Let Y an object of M. A simplicial resolution of Y is a simplicial object over M noted Y_* together with a weak equivalence $Y \to Y_0$ such that

1. the object Y_0 is fibrant
2. all faces maps in Y_* are acyclic fibrations. Hence, all the objects Y_n are fibrants.
3. Let (d_*, Y_n) the diagram:
   (a) for every $0 \leq i \leq n+1$, a copy $(d_i; Y_n)$ of Y_n
   (b) for every $0 \leq i < j \leq n+1$, a copy $(d_i d_j, Y_{n-1})$ of Y_{n-1}
   (c) pair of maps: $(d_j, Y_n) \to (d_i d_j, Y_{n-1}) \leftarrow (d_j, Y_n)$. These arrows are acyclic fibrations.

   Then the map $Y_{n+1} \to (d_*, Y_n)$ is a fibration.

Definition 7.2. A simplicial resolution Y_* of Y can be viewed as a simplicial object over M together with a simplicial map $i : Y \to Y_*$ (here Y represents the constant simplicial object over M made from the object Y), and having some further properties. Then one can define a map of simplicial resolutions of Y, $f : Y_* \to Y'_*$ as a simplicial map which makes the following diagram commute:



Definition 7.3. The notion of cosimplicial resolution and map between cosimplicial resolutions are defined dually from the notion of simplicial resolution and map of simplicial resolutions.

With these notions, one can try to state the definition:

Definition 7.4. $hom_M(X^*, Y_*)$ has an obvious structure of bisimplicial set for X^* a cosimplicial resolution of X and Y_* a simplicial resolution of Y. The homotopy complex or mapping space of (X, Y) is then defined (up to homotopy) as $map(X,Y) := diaghom_M(X^*, Y_*)$.

However, one needs some machinery to make this definitions coherent. Indeed one has to check that the homotopy can be defined for two objects (X, Y) and that it does not depens on the resolutions chosed. The first requirement is answered by:

Proposition 7.1. *Every object of a model category M has simplicial and a cosimplicial resolutions.*

Proof. Let's show the result by induction for a simplicial resolution. One adds a requirement of this simplicial resolution (usefull for the construction): the degeneracies are acyclic cofibrations.

- If $*$ is the final object of M, then $Y \to *$ can be factorized through : $Y \overset{\sim}{\hookrightarrow} Y_0 \twoheadrightarrow *$. Then one got the fibrant element Y_0 and an acyclic fibration from Y to Y_0.
- If one has the elements Y_i for $0 \leq i \leq n$ with the corresponding faces and degeneracies maps which are respectively acyclic fibrations and acyclic cofibrations, then one defines (s_*, Y_n) as the direct limits of the diagram made of copies (s_i, Y_n) of Y_n for $0 \leq i \leq n$ and copies $(s_i s_j, Y_{n-1})$ of Y_{n-1} for $0 \leq i < j \leq n$ together with maps $(s_i, Y_n) \xleftarrow[s_{j-1}]{} (s_i s_j, Y_{n-1}) \xhookrightarrow{s_i} (s_j, Y_n)$. Then one has easily a weak equivalences $(s_*, Y_n) \xrightarrow{\sim} (d_*, Y_n)$ which can be factorized through $(s_*, Y_n) \overset{\sim}{\hookrightarrow} Yn+1 \overset{\sim}{\twoheadrightarrow} (d_*, Y_n)$.

The cosimplicial resolution is created dually. □

Remark 7.1. In the proof we had added a property: the map $Y \to Y_0$ and the degeneracies are acyclic cofibrations. Such a simplicial resolution is called cofibrant. We define dually the notion of fibrant cosimplicial resolution. The proof of the previous property shows that every object of M has cofibrant simplicial resolutions and fibrant cosimplicial resolutions.

The next lemma uses this last notion to make links between the various homotopy function complexes one can construct.

Lemma 7.1. $[DK3, 3, 6.9 and 6.10]$

1. *Let Y_* and Y'_* be simplicial resolutions of Y. Then, if Y'_* is cofibrant, there exists a map of resolutions $Y_* \to Y'_*$.*
2. *Let X^* and X'^* be cosimplicial resolutions of X. Then, if X'^* is fibrant, there exists a map of resolutions $X'^* \to X^*$*

Sketch of proof. The maps are constructed by induction □

Proposition 7.2. $[DK3, 4, 17.3.4]$ *If $f : X'^* \to X^*$ is a map of cosimplicial resolutions of X and $g : Y_* \to Y'_*$ is a map of simplicial resolutions of Y, then they induced a map of simplicial set $diaghom_M(X^*, Y_*) \to diaghom_M(X'^*, Y'_*)$ which is a weak homotopy equivalence*

Corollary 7.1. *One can find a finite string of weak homotopy equivalences between two homotopy function complexes constructed for the pair of objects (X, Y). Hence, the homotopy function complex is unique up to weak homotopy equivalence.*

The next proposition allows us to link the homotopy function complexes to the usual notion of homotopy in a model category:

Proposition 7.3.

1. *If X^* is cosimplicial resolution in M, then $X^0 \sqcup X^0 \to X^1 \to X^0$ is a cylinder object of X^0.*
2. *If Y_* is simplicial resolution in M, then $Y_0 \to Y_1 \to Y_0 \times Y_0$ is a path object of Y_0.*

Then, a cosimplicial resolution of X is a sort of "higher cylinder objects" collection for a cofibrant approxiation of X. Dually, a simplicial resolution of Y is a sort of "higher path objects" collection for a fibrant approximation of Y.

8 Bousfield localization

Let M be a model category and $\mathcal{S}$ a set of arrows. If we want to localize $Ho(\mathrm{M})$ with respect to $\mathcal{S}$, one solution which does not loose the higher homotopical data is to work directly on M by growing the set of weak equivalences. It is what the Bousfield localization do.

Definition 8.1. A left localization of M with respect to $\mathcal{S}$ is a pair $(L_{\mathcal{S}}\mathrm{M}, j)$ universal among the pairs (N, ϕ) where N is a model category and $\phi : \mathrm{M} \to \mathrm{N}$ is a left Quillen functor (preserves cofibrations and acyclic cofibrations left adjoint of an adjunction) such that its total derived functor sends $\mathcal{S}$ to isomorphisms.

We will see in what cases we can build a left Bousfield localization.

Definition 8.2. An object A of M is (left) $\mathcal{S}$-local if it is fibrant and for every $X \to Y$, the induced simplicial morphism $Map(Y, A) \to Map(X, A)$ is a weak homotopy equivalence.
A morphism $f : A \to B$ of M is a (left) $\mathcal{S}$-local equivalence if for every $\mathcal{S}$-local object X, the induced map $Map(B, X) \to Map(A, X)$ is a weak homotopy equivalence.

Definition 8.3. A left Bousfield localization of M with respect to $\mathcal{S}$ is the data of a model structure $\mathtt{L}_{\mathcal{S}}\mathrm{M}$ on the underlying category of M such that:

1. the weak equivelences are the $\mathcal{S}$-local equivalences
2. the cofibrations are the cofibrations of M

In that case, the canonical functor $\mathrm{M} \to \mathtt{L}_{\mathcal{S}}\mathrm{M}$ is left Quillen and it is a left localization of M with respect to $\mathcal{S}$ [Hi3,3.3.19].

Proposition 8.1 (Hi,4.1.1)**.** *A left proper cellular model category has left Bousfield localizations which is left proper cellular, whom fibrant objects are the $\mathcal{S}$-local objects.*

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